20071207 Seiberg-Witten curve

$M(k,r) \xrightarrow{\pi} \overline{M}(k,r) \quad \curvearrowleft \quad \widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T^{r-1}$

Gieseker $\qquad$ Uhlenbeck

$$[\overline{M}(k,r)] = \pi_* [M(k,r)] \in H^{\widetilde{T}}_*(\overline{M}(k,r))$$

$$S(\tilde{t}^*) = H^*_{\widetilde{T}}(pt) = S(\mathrm{Lie}\,\widetilde{T}^*) = \mathbb{C}[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]$$

$\vec{a} = (a_1, \dots, a_r)$ $\qquad$ $(a_1 + \dots + a_r = 0)$

$\mathcal{S}(\tilde{t}^*)$ = quotient field

$$H^{\widetilde{T}}_*(\overline{M}(k,r)) \otimes_{S(\tilde{t}^*)} \mathcal{S}(\tilde{t}^*) \overset{\iota_*}{\cong} H^{\widetilde{T}}_*(\overline{M}(k,r)^{\widetilde{T}}) \otimes_{S(\tilde{t}^*)} \mathcal{S}(\tilde{t}^*) = \mathcal{S}(\tilde{t}^*)$$

$$Z^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) \overset{\text{def.}}{=} \sum_{k=0}^{\infty} \Lambda^{2kr} (\iota_*)^{-1} [\overline{M}(k,r)]$$

$$\in \mathbb{Q}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$$

<u>Rem</u> There are K-theoretic version $\leftarrow$ 5D (in phy)

elliptic genus version $\leftarrow$ 6D

Nekrasov's conjecture

1) $\varepsilon_1\varepsilon_2 \log Z^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$ is regular at $(\varepsilon_1, \varepsilon_2) = (0,0)$

2) $F^{inst}(\vec{a}; \Lambda) = \varepsilon_1\varepsilon_2 \log Z^{inst}\Big|_{\varepsilon_1=\varepsilon_2=0}$

is the instanton part of the Seiberg-Witten prepotential.

Comments

① After localizing only $\vec{a}$,

not localized!

$$H^*_{\mathbb{C}^*\times\mathbb{C}^*}(\ )\otimes\mathbb{C}(\vec{a})$$

$$Z^{inst} = \sum \Lambda^{2\sum n_\alpha} \int_{\prod_{\alpha=1}^{r} \mathrm{Hilb}^{n_\alpha}} \cdots\cdots = \sum \Lambda^{2\sum n_\alpha} \int_{\prod S^{n_\alpha}\mathbb{C}^2} \cdots\cdots$$

As $\sum_{n=0}^{\infty} S^n\mathbb{C}^2 = \sum \frac{1}{n!}(\mathbb{C}^2)^n = $ "$\exp \mathbb{C}^2$",

we expect above $= \exp \int_{\underbrace{(\mathbb{C}^2 \sqcup \mathbb{C}^2 \sqcup \cdots \sqcup \mathbb{C}^2)}_{r \text{ copies}}} \cdots\cdots$

Applying the localization of $\varepsilon_1, \varepsilon_2$ at this state we only get the pole $\frac{1}{\varepsilon_1\varepsilon_2}$, i.e. Conj. 1)

(This is easy to see $r=1$ case

$$\int_{S^n\mathbb{C}^2} 1 = \frac{1}{n!}\int_{\mathbb{C}^{2n}} 1 = \frac{1}{n!(\varepsilon_1\varepsilon_2)^n} \quad \therefore Z = \exp\left(\frac{1}{\varepsilon_1\varepsilon_2}\right)$$

① Geometric engineering

$\log Z^{inst}(\varepsilon_1, -\varepsilon_1, \vec{a}; \Lambda)$ = limit of GW inv.

$$= \lim_{\substack{\text{K-theory} \\ \to \text{cohomology}}} \left(\sum_{\substack{g \geq 0, \\ d > 0}} \varepsilon_1^{2g-2} \Lambda^d \, \#\{f : \Sigma_g \to \begin{smallmatrix}\text{noncpt} \\ \text{toric} \\ \text{CY}\end{smallmatrix}\} \right)$$

Thus (at least after $\varepsilon_1 = -\varepsilon_2$)

1) $\Leftarrow$ genus ≥ 0

2) SW prepotential = genus 0 GW inv. of noncpt CY

② Mirror symmetric picture

A-model	B-model	
instanton moduli space	SW curve	
instanton counting	period integral	$\int_{A_\alpha} d(SW)$, $\int_{B_\alpha} d(SW)$ SW : differential
genus 0 GW inv's of noncpt CY M	$\int_{A_\alpha, B_\alpha} \Omega_{M^\vee}$	$\Omega_{M^\vee}$: holo 3-form $M^\vee$ = mirror of M

$$\{zw = H(p,x)\} = M^\vee$$
$$\downarrow \quad \mathbb{C}^*\text{-fibration} \quad \text{degenerate at}$$
$$(p,x) \in \mathbb{C}^2 \qquad \supset \text{SW curve } H(p,x)=0$$

$$\Omega = \frac{dz \wedge dp \wedge dx}{z}$$

$\rightsquigarrow$ integral is reduced to the integral over SW curve

As in the case of the mirror symmetry,
the relation between A-model & B-model
are mysterious, even though we can
give a rigorous proof.

In particular, we cannot see the SW curves
in the classical gauge theory. It is
truely a quantum concept.

Plan ① perturb/instanton parts
② definition of the SW prepotential
(mathematically rigorous, see 0311058)
③ SW's intuition
Why it is expected to be related to
the gauge theory?

It is an indirect argument
even in a physical level of rigor.

Next week : A rigorous proof $+ \alpha$

① perturb. / instanton parts

SW prepotential has the classical, perturbative & instanton parts.

The classical & perturbative parts are explicit functions, independent of the instanton calculus.

classical

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \overset{\text{def.}}{=} \frac{1}{\varepsilon_1\varepsilon_2}\left\{\frac{3}{4}x^2 - \frac{1}{2}x^2\log\left(\frac{x}{\Lambda}\right)\right\} + \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\left\{x - x\log\frac{x}{\Lambda}\right\} - \frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log\left(\frac{x}{\Lambda}\right) + \sum_{n\geq 3}\frac{c_n(\varepsilon_1,\varepsilon_2)\,x^{n-2}}{n(n-1)(n-2)}$$

$$\left(\frac{t^2}{(1-e^{\varepsilon_1 t})(1-e^{\varepsilon_2 t})} = \sum_{n\geq 0} c_n(\varepsilon_1,\varepsilon_2)\,t^n\right)$$

($\rightsquigarrow$ $n-2$ will become $2g-2$; $n\geq 3$ $2g\geq 3$ $g\geq 2$)

Then we define

$$Z = \exp\left[-\sum_{\alpha\neq\beta}\gamma_{\varepsilon_1,\varepsilon_2}(a_\alpha - a_\beta;\Lambda)\right] Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)$$

branch of log must be fixed carefully, but we ignore it in this lecture.

some region $\supset$ (a_α: real $a_1 \ll a_2 \ll \cdots \ll a_r$)

This becomes clear in the SW prep.

Explanation : x ---- $c_1(L)$ L: equiv. line b'dle on $\mathbb{C}^2$

$\varepsilon_1, \varepsilon_2$ ---- Chern roots of $T\mathbb{C}^2$

$c_1(T\mathbb{C}^2) = \varepsilon_1 + \varepsilon_2 \quad c_2(T\mathbb{C}^2) = \varepsilon_1\varepsilon_2$

The above is computing some integral over $\mathbb{C}^2$.
equiv.

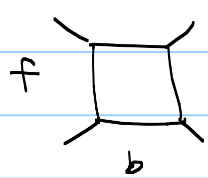
Wallcrossing formula for proj. surfaces $\rightsquigarrow$ int. over X
The above formula naturally come up.

- $$\frac{\varepsilon_1\varepsilon_2}{(1-e^{\varepsilon_1})(1-e^{\varepsilon_2})}$$

  is the Todd genus of $\mathbb{C}^2$.

  This appears (in the above computation) to compute "dim $H^*(X, L)$"

- In the geometric engineering (GW on non cpt CY) the perturb. part has a similar structure. But the perturb. in gauge theory contains the instanton part in GW.



Λ --- $e^{-[\text{base}]}$

i.e., GW inv. with degree to base $= 0$

SW curve

$\vec{u} = (u_2, u_3, \cdots, u_r) \in \mathbb{C}^{r-1}$: parameter

Λ $\quad$ $\subset$ "u-plane"

$P(z) = z^r + u_2 z^{r-2} + \cdots + u_r$: polynomial

$$C_{\vec{u}} : \quad y^2 = P(z)^2 - 4\Lambda^{2r}$$
$$= (P(z) - 2\Lambda^r)(P(z) + 2\Lambda^r)$$
$$\downarrow$$
$$\mathbb{P}^1 \ni z$$

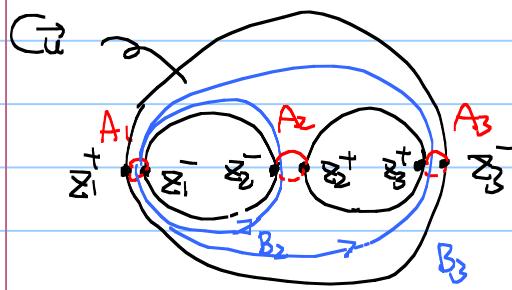
hyper-elliptic curves parametrised by $\vec{u}$

Suppose $\quad P(z) = \prod_{\alpha=1}^{r} (z - z_\alpha)$

$\Lambda = 0$ ---- "classical limit"

We consider the region, where $|\Lambda|$ is sufficiently smaller than $|z_\alpha - z_\beta|$.

$z_\alpha^\pm$: $\quad P(z_\alpha^\pm) = \pm 2\Lambda^r \quad$ & $\quad z_\alpha^\pm \doteqdot z_\alpha$



↺ hyperelliptic involution
$y \mapsto -y$

We choose cycles A_α, B_α as above.

(We only move in a small region, so that we don't need to consider the monodromy. But in the original SW paper, the monodromy plays a crusial role.

<u>SW differential</u>

$$dS = -\frac{1}{2\pi}\frac{zP'(z)}{y}dz$$

mero. diff. poles at $\infty_\pm$

$$a_\alpha \equiv a_\alpha(u) = \int_{A_\alpha} dS \,, \quad a_\alpha^D = \int_{B_\alpha} dS$$

$$\frac{\partial a_\alpha}{\partial u_p} = \frac{1}{2\pi}\int_{A_\alpha}\frac{z^{r-p}dz}{y}$$

basis of holo diff. of the hyperelliptic curve $C_{\vec{u}}$

Thus this is invertible.

$$\therefore (\tau_{\alpha\beta}) = \left(\frac{\partial a_\beta^D}{\partial u_p}\right)\left(\frac{\partial a_\alpha}{\partial u_p}\right)^{-1} \quad ; \text{ period matrix of } C_{\vec{u}}$$

$$= \left(\frac{\partial a_\beta^D}{\partial a_\alpha}\right)$$

symmetric

We now switch u_p & a_α.

Consider a_α : variable u_p : function in a_α

$\therefore$ $\exists$ locally defined holo. func. $\mathcal{F}_1$

s.t. $$a_\alpha^D = -\frac{1}{2\pi i}\frac{\partial \mathcal{F}_1}{\partial a_\alpha}$$

This $\mathcal{F}_1$ is called the Seiberg-Witten prepotential.

Prop. $$\mathcal{F}_1 = -\sum_{\alpha \neq \beta} \gamma_0(a_\alpha - a_\beta ; \Lambda) + O(\Lambda^{2r})$$

instanton part!

where $$\gamma_0(x;\Lambda) = \varepsilon_1\varepsilon_2 \gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)\Big|_{\varepsilon_1=\varepsilon_2=0}$$

$$= \frac{3}{4}x^2 - \frac{1}{2}x^2 \log\frac{x}{\Lambda}$$

Th. $\quad \varepsilon_1\varepsilon_2 \log Z = \mathcal{F}_1 + \text{higher}$

There are several proofs
N-Yoshioka
Nekrasov-Okounkov, Braverman-Etingof

A physical explanation

Assume $SU(2)$ for simplicity.

$F(a,\Lambda) \overset{\text{def}}{=} \varepsilon_1\varepsilon_2 \log Z(\varepsilon_1,\varepsilon_2,a,\Lambda)\Big|_{\varepsilon_1=\varepsilon_2=0}$ (want to show $= \mathcal{F}_1$)

$F \in \mathbb{C}(a)[[\Lambda]]$ (In fact, $a \leftrightarrow -a$)

Recall a is the coordinate on

$$\operatorname{Spec} H_T^*(pt) = \operatorname{Spec} \mathbb{C}[a].$$

In fact, it is more natural to consider

$$H^*_{SU(2)}(pt) = H_T^*(pt)^W = \mathbb{C}[a_1, a_2]^{S_2}_{a_1+a_2=0}$$

This is called "the space of vacua"
(or classical moduli space) in the physics literature.

$\operatorname{Spec} H^*_{SU(2)}(pt) = \mathfrak{g}_{\mathbb{C}} / W$: u-plane

$\left(= \{ \varphi \in \mathfrak{g}_{\mathbb{C}} \mid [\varphi, \varphi^\dagger] = 0 \} / G \right)$

in Diaconescu's lecture

singular at 0

It is more natural to consider classical $u = a^2$
(in general, p^{th} elementary symm. poly. in $\vec{a}$
coord. of classical u-plane.

Then $F^{\text{classical}}$ is
a "potential" (in the special Kähler geometry).

"metric" $\dfrac{\partial^2 F^{\text{class}}}{\partial a^2} = \text{constant}$

Our F is the quantum correction to $F^{\text{class.}}$.

In particular, the cpx str. of u-plane
is also corrected!

In fact, it is easier to see this phenomenon in $u = -\frac{1}{4}\frac{\partial}{\partial \log \Lambda} F$

(This can be proved by the SW-side).

And if we go back to the definition of $F = \varepsilon_1 \varepsilon_2 \log Z |_{\varepsilon_1=\varepsilon_2=0}$, then we find

$$☆ \quad u = \lim_{\varepsilon_1, \varepsilon_2 \downarrow 0} \frac{1}{Z} \sum \Lambda^{4k} \int_{M(2,k)} ch_2(\mathcal{E})/[0]$$

$$= \text{"} \langle ch_2(\mathcal{E})/[0] \rangle \text{"} \quad \text{(correlation function)}$$

where $\mathcal{E}$: universal sheaf on $\mathbb{C}^2 \times M(2,r)$

classical part = $(k=0)$ part

$$\mathcal{E} = e^a \oplus e^{-a} \Rightarrow ch_2 = a^2 = u^{\text{classical}}$$

Thus u is the quantum correction of $u^{\text{classical}}$.

"proof" of ☆ : we compute via the fixed pt localization

$$ch_2(\mathcal{E})\Big|_{\substack{\mathcal{E} = I_1 \oplus I_2 \\ \uparrow \quad \uparrow \\ \text{ideal sheaf}}} = ch_2(e^a I_1 \oplus e^{-a} I_2)$$

$$= a^2 - \underbrace{(\text{length } Z_1 + \text{length } Z_2)}_{\substack{\| \\ \text{instanton } \#}}$$

$$0 \to I_1 \to \mathcal{O} \to \mathcal{O}_{Z_1} \to 0$$

$\frac{\partial}{\partial \log \Lambda}$: counts the instanton # //

Q. How to compute $\mathcal{F}$?

SW used several physical insights. $\dfrac{a}{\Lambda}$ ... fixed

0 If $|a|$ is sufficiently large (Λ is small by homogeneity),

$\mathcal{F}$ is approximated by $\mathcal{F}^{\text{class.}}$

(The gauge theory is asymptotically free.)

This was used to compute the monodromy at $u=\infty$

1 The low energy effective theory is described as a $U(1)$-gauge theory.

For each a, $\exists$ a SUSY $U(1)$-gauge theory.

($U(1)$ comes from $Z_G(a) \cong U(1)$ for $a \neq 0$.)

cf. Bosonic $U(1)$-gauge theory $\bar{\tau}\int \|F_+\|^2 - \tau \int \|F_-\|^2$

In our case $\tau = \dfrac{1}{2\pi\sqrt{-1}} \dfrac{d^2 \mathcal{F}}{da^2}$.

So the coupling constant varies as we move a.

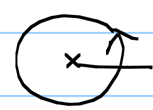
We require $\operatorname{Im}\tau > 0$.

Moreover

elect., magn. charges

The $U(1)$-gauge theory has the charge lattice $\mathbb{Z}^2$.
So the special Kähler geometry should
comes from the integrable system,
i.e. a family of elliptic curves parametrised by u
(by Freed's result)

② Recall that $U(1)$-gauge theory has the
electro/magnetic duality changing
$$a \longleftrightarrow a_D = \frac{\partial F}{\partial a}, \quad \tau \longleftrightarrow -1/\tau .$$
There is no reason to prefer a from a_D or more general linear comb.

When we analytically continue a, a_D around
the singularity e.g.  $u=\infty$
we come back to the same $U(1)$ gauge theory,
but possibly with different a, a_D
under $SL(2,\mathbb{Z})$.

From ⓪, we can compute
the monodromy at $u=\infty$
and find
$$M_\infty = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}.$$

③ Understand other singularities:

In Classical moduli, the other singularity is at $u=0$.

$\longleftrightarrow Z_G(a) = SU(2) \supsetneq U(1)$

So we need to go back to the $SU(2)$ gauge theory from $U(1)$-theory.

But this is not true any more in the quantum level.

contradicts with $\mathrm{Im}\,\frac{\partial^2 F}{\partial a^2} > 0$

At singularities, BPS states becomes massless.

$$m = |\underset{\mathbb{Z}}{Q_e}\, a + \underset{\mathbb{Z}}{Q_m}\, a_D|$$

$\therefore$ some cycle $\subset$ torus collapse:

$\longleftrightarrow$ discriminant of the family of elliptic curves

Assume 2 singular points

- $a_D = 0 \Rightarrow$ monopole becomes massless

  $\left(\int_B \omega = 0\right)$

- $a + a_D = 0 \Rightarrow$ dyon massless

After computing monodromy,

SW gave the family of elliptic curve, satisfying these requirements.